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Automata Simulating Quantum Logics

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The idea of computational complementarity is developed further. A special class of macroscopic automata to imitate quantum and classical systems is described. The simplest automaton imitating a spin-1/2 particle is completely considered.

1. INTRODUCTION

Niels Bohr was the first to think that the discovery of quantum mechanics was something more than that of new laws of microphysics; it was a new point of view which could be useful in different areas, for example, biology or economics.

The aim of this paper is to find some examples which have nothing to do with microphysics but use the same mathematical formalism as quantum mechanics. These examples can be found in economics, sociology, and theory of automata. The possibility of constructing macroscopic automata imitating quantum systems is very important for the general problem of imitating the modeling of microphysical processes and constructing special computers for this aim.

The main idea is to find "different representations" of quantum logics. By quantum logic we mean, following Birkhoff and von Neumann (1936), some nondistributive lattice which corresponds on one hand to some quantum microsystem and on the other to some classical system. D. Finkelstein was the first to see the correspondence between quantum lattices and graphs (Finkelstein and Finkelstein, 1983) which makes it possible to find macroscopic realizations of quantum logics. To illustrate the idea, consider a very simple quantum system: a particle with spin one-half which is described by two projections of spin S_z and S_x. The lattice of properties of this particle

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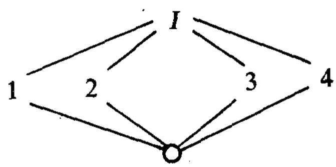
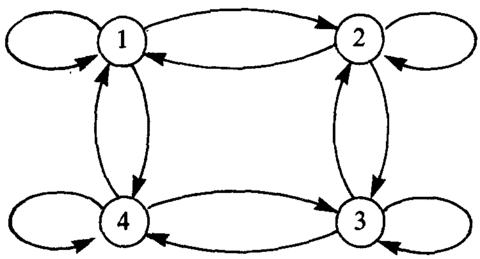
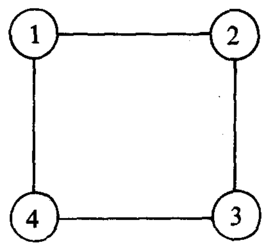
is an orthomodular nondistributive lattice:



To this lattice there corresponds the graph (Finkelstein and Finkelstein, 1983).



or, briefly,



Now consider the opposite question: in what sense does the nondistributive lattice correspond to the graph? Let 1, 2, 3, 4 be states of some system (e.g., an economic one) and suppose that there is an observer who tries to check the state of the system by putting questions to it. The system has the following property: it can answer "yes" to the question "are you in 2" not only if it is in 2, but also if it is in 1 or 3. It can change its state by one step responding to the question if and only if corresponding states are connected by an arc. But let the observer be clever enough to know this property of the system: then he must use some "negative logics." He concludes that the system is in 2 if to the question "are you in 4?" a negative answer is obtained. So by a negative answer to a complementary question he can know the real state of the system. But then it is easy to see that one can find no such questions the negative answer to which corresponds to the state "1 or 2," "2 or 3," and so on. This means that in our "negative logics," "1 or 2" coincides with I "any state." One cannot find any difference between disjunctions $1 \vee 2, 2 \vee 3, 3 \vee 4, 1 \vee 4$, and I. That is why the lattice is nondistributive. $1 \vee 2$ is true if 1 is true, $1 \vee 2$ is true if 2 is true, but not only if: $1 \vee 2$ can be true when both 1 and 2 are false.

It is easy to see that our observer cannot use probability theory for the system he controls, because there is no probability measure for a nondistributive lattice. For example, in the symmetrical case, the probability of each state must be $1/4$. But because $1 \vee 2 = I$, the probability of $1 \vee 2$ must be equal to 1, but it is equal to $1/4 + 1/4 = 1/2$. In the following sections we give some rigorous results for constructing macroscopic realizations of quantum logics.

Nevertheless we must make an important remark. Surely we agree with Jauch (1968) and D'Espagnat (1976) that quantum logics do not represent an alternative to the Copenhagen interpretation and there are properties of composite systems inconsistent with Bell's inequalities. These properties cannot be imitated by usual physical processes in macrosystems. Now, we have no clear answer to whether the violation of Bell's inequalities is an obstacle that prevents the imitation of all quantum properties by macroscopic automata. Here our aim is more modest: to imitate only some quantum properties.

2. NORMALIZED AUTOMATA AND THEIR LATTICES

Consider first a classical system with a finite number of states. Let us represent it as fully disconnected graph:

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1 2 N

In this case the set $1, 2, \ldots, N$ is a phase space of the system. The Boolean lattice 2^N of all subsets of the phase space is a property lattice of the system, which can be built step by step:

Ground story: empty set $\varnothing$.
First story: consists of singletons $1, \ldots, N$.
Second story: consists of two-element sets 12; 13; ...; $N-1, N$.
...
Nth story: the set $1, 2, \ldots, N$ itself.

The lattice is self-dual: if one turns it over and exchanges each element by its complement to $1, 2, \ldots, N$, we again obtain the same lattice 2^N.

For imitating classical and quantum systems we shall use normalized automata.

Definition. A normalized automaton A is an automaton defined by a nonoriented graph G so that:

1. The set of input symbols and the set of interior states of A coincide with $V(G)$, the set of vertices of the graph.

2. The transition function is defined in the following way. If the automaton is initially in the state i and the input symbol is j, then, if the vertices i and j are adjacent, i.e., connected with an arc, the new state will be j, and if they are not connected, it stops its work:

$$(i, j) = \begin{cases} j & \text{if} \quad (i, j) \in \mathrm{Arc}(G) \text{ (set of arcs of } G) \\ \mathrm{STOP} & \text{otherwise} \end{cases}$$

Each vertex is by definition adjacent to itself: the graph is reflexive.

To each nonoriented graph considered as a normalized automaton we can put into correspondence a lattice of its observed properties by the following algorithm:

1. At the foundation: the empty set.
2. The first story consists of stars of each vertex. The *star* $[i]$ of a vertex i is the set of all vertices adjacent to i.
3. Then construct all set-theoretic unions of stars.
4. Now turn the obtained lattice over and exchange each element by its set-theoretic complement.

This algorithm is a simplification of that proposed in Finkelstein and Finkelstein (1983) for the case of normalized automata. Now consider some examples.

Example 1. Nonconnected graph with N vertices. As was shown above, it generates a Boolean lattice 2^N.

Example 2. Measurement of two projections of spin 1/2 (see Figure 1).

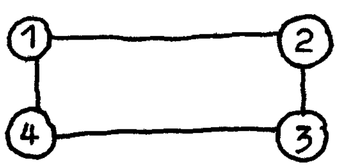
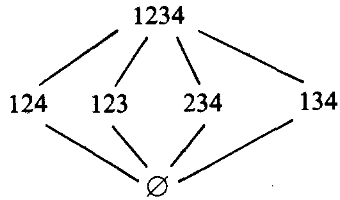


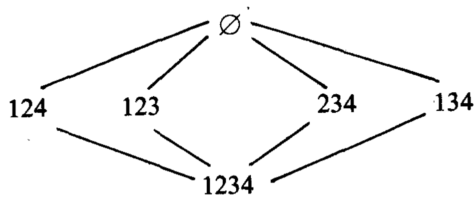
Fig. 1. The graph generating the lattice M_4. Ground story, $\varnothing$.

The first story consists of four elements: stars of vertices 1-4: $[1] = 124$, $[2] = 123$, $[3] = 234$, $[4] = 134$.

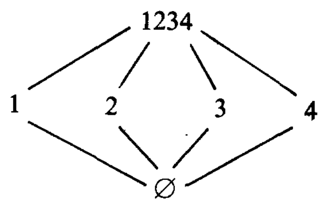
The union of any two elements of first story gives the set $V(G) = 1234$; thus, the lattice has the form



Turn it over,

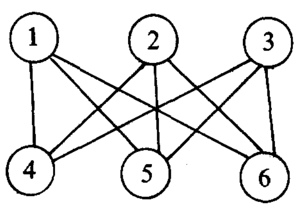


Replace each element by its complement,



This lattice is not distributive but modular: $1 \wedge (2 \vee 3) \neq (1 \wedge 2) \vee (1 \wedge 3)$.

Example 3. A graph generating Gudder's (1983) quark model lattice:

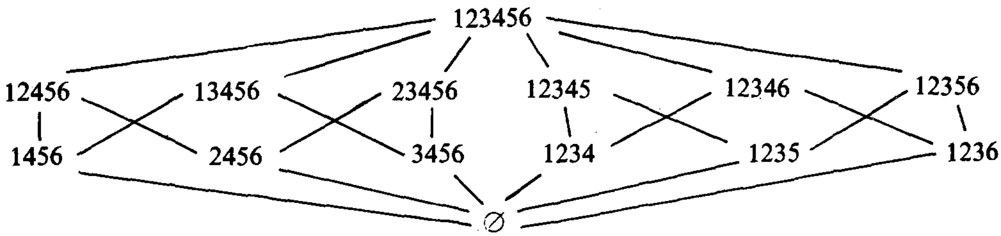


Ground story: ∅.
First story:

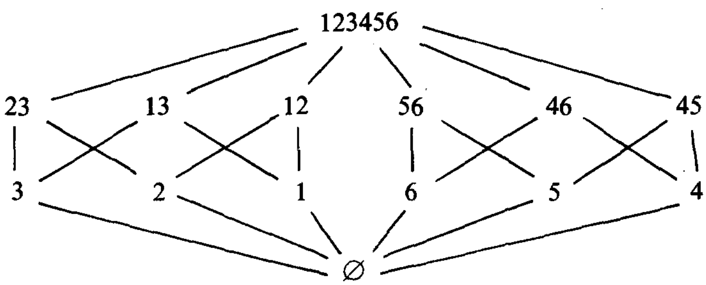
$$[1] = 1456, \quad [2] = 2456, \quad [3] = 3456$$

$$[4] = 1234, \quad [5] = 1235, \quad [6] = 1236$$

The lattice of unions has the form

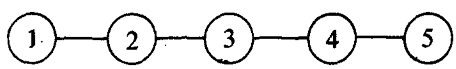


Turn it over and replace its elements by complements:

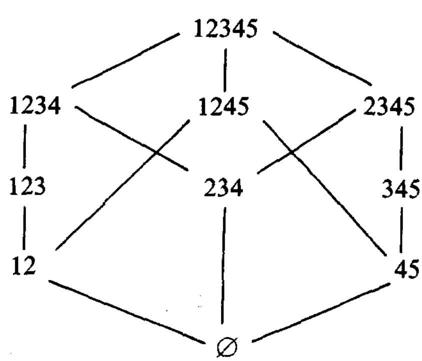


The lattice is also not distributive but modular: $1 \wedge (2 \vee 6) \neq (1 \wedge 2) \vee (1 \wedge 6)$, but $1 \wedge (2 \vee 3) = (1 \wedge 2) \vee (1 \wedge 3)$.

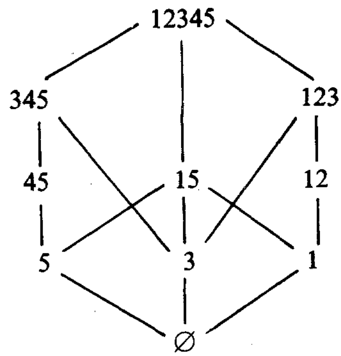
Example 4. Nonmodular lattices can also be obtained. Consider the graph



Its stars are $[1] = 12$, $[2] = 123$, $[3] = 234$, $[4] = 345$, $[5] = 45$. The union lattice is



The property lattice is



In conclusion, we adduce two theorems allowing us to construct new lattices from given ones.

Let G_1, G_2 be two arbitrary graphs and $L(G_1)$, $L(G_2)$ their property lattices constructed in accordance with the algorithm described above.

Theorem 1. If G is a graph obtained by placing together two graphs G_1 and G_2,

$$G = G_1 + G_2$$

then $L(G) = L(G_1) \times L(G_2)$, a lattice product (Birkhoff, 1967).

Theorem 2. If G is a bunch of G_1 and G_2, i.e., each vertex of G_1 is connected by an arc with each vertex of G_2,

$$G = G_1 \triangle G_2$$

then $L(G) = L(G_1) + L(G_2)$, a horizontal sum (Birkhoff, 1967).

The proof of Theorems 1 and 2 is in Zapatrin (1988).

Remark. The algorithm cited above for constructing the lattice by a graph can be reversed. In this case, for a given finite lattice one can built the graph generating this lattice (Zapatrin, 1988).

3. LOGICAL DESCRIPTION OF SPIN-1/2 PARTICLE

Consider the graph of Figure 2. Its vertices correspond to the following questions:

1: "$S_x = +1/2$?"	4: "$S_x = -1/2$?"
2: "$S_y = +1/2$?"	5: "$S_y = -1/2$?"
3: "$S_z = +1/2$?"	6: "$S_z = -1/2$?"

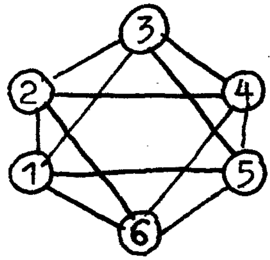


Fig. 2. Complete graph for spin-1/2 particle.

In the usual Hilbert space formalism these questions correspond to projectors onto the state vectors:

$$\begin{aligned} e_1 &= |x+\rangle = 1/\sqrt{2}\begin{pmatrix}1\\1\end{pmatrix}\\ e_2 &= |y+\rangle = 1/\sqrt{2}\begin{pmatrix}i\\-1\end{pmatrix}\\ e_3 &= |z+\rangle = \begin{pmatrix}1\\0\end{pmatrix}\\ e_4 &= |x-\rangle = 1/\sqrt{2}\begin{pmatrix}1\\-1\end{pmatrix}\\ e_5 &= |y-\rangle = 1/\sqrt{2}\begin{pmatrix}i\\1\end{pmatrix}\\ e_6 &= |z-\rangle = \begin{pmatrix}0\\1\end{pmatrix}\end{aligned} \tag{3.1}$$

Let us now describe pure states in terms of weights on the graph. To any unit vector $\psi \in C^2$ we shall put into correspondence a set of weights a_α. The physical meaning of a_α is the probability of an affirmative answer to a question corresponding to the vertex α. Thus, $a_\alpha = |P_\alpha \psi|^2$. In detail, if

$$\psi = \begin{pmatrix} r_1\, e^{i\phi_1} \\ r_2\, e^{i\phi_2}\end{pmatrix}, \qquad r_1^2 + r_2^2 = 1$$

then, denoting $\phi_{12} = \phi_2 - \phi_1$,

$$\begin{aligned} a_1 &= 1/2 + r_1 r_2 \cos \phi_{12} & a_4 &= 1/2 - r_1 r_2 \cos \phi_{12}\\ a_2 &= 1/2 + r_1 r_2 \sin \phi_{12} & a_5 &= 1/2 - r_1 r_2 \sin \phi_{12}\\ a_3 &= r_1^2 & a_6 &= r_2^2 \end{aligned} \tag{3.2}$$

Since (3.2) contain only differences of the phases $\phi_2-\phi_1$, the state vector can be restored only up to a phase multiple.

For two given pure states

$$\psi=\begin{pmatrix} r_1\, e^{i\phi_1} \\ r_2\, e^{i\phi_2} \end{pmatrix}, \qquad \phi=\begin{pmatrix} s_1\, e^{i\theta_1} \\ s_2\, e^{i\theta_2} \end{pmatrix}$$

the transition probability is

$$p=|\langle\psi\mid\phi\rangle|^2=r_1^2s_1^2+r_2^2s_2^2+2r_1r_2s_1s_2\cos(\phi_1-\theta_1-\phi_2+\theta_2)$$

Let

$$a_\alpha=|P_\alpha\psi|^2, \qquad a'_\alpha=|P_\alpha\phi|^2, \qquad \alpha=1,\ldots,6$$

Then, in accordance with (3.1), we obtain

$$P=\sum_{\alpha=1}^{6} a_\alpha a'_\alpha-1$$

Corollary. For any pure state the transition probability into itself is always 1; therefore, for any set of weights a describing a pure state we always have

$$\sum_{\alpha=1}^{6} a_\alpha^2=2 \tag{3.3}$$

4. DESCRIPTION OF OBSERVABLES

In the usual formalism the following self-adjoint operator in C^2 corresponds to an observable:

$$A=\begin{pmatrix} a & b+ci \\ b-ci & d \end{pmatrix}$$

where a, b, c, and d are arbitrary real parameters.

The quantum logic approach proposes to set an observable by its mean values in fixed pure states. In our case these are

$$A_\alpha=\mathrm{Tr}(P_\alpha A), \qquad \alpha=1,\ldots,6$$

where P_α are projectors on given vectors $e_1,\ldots,e_6$.

A direct calculation yields

$$\begin{aligned} 2A_1&=a+2b+d & 2A_4&=a-2b+d \\ 2A_2&=a-2c+d & 2A_5&=a+2c+d \\ A_3&=a & A_6&=d \end{aligned} \tag{4.1}$$

Thus, we can restore the parameters of the operator by the following formulas:

$$a = A_3, \qquad b = \frac{A_1 - A_4}{2}, \qquad c = \frac{A_5 - A_2}{2}, \qquad d = A_6$$

The trace $\bar{A}$ of operator A is equal to

$$\bar{A} = A_1 + A_4 = A_2 + A_5 = A_3 + A_6 \tag{4.2}$$

In the traditional formalism the mean value of A in state ψ is $EA = \langle\psi|A|\psi\rangle$. In terms of weights the calculation gives

$$E_\psi A = \sum_{\alpha=1}^{6} A_\alpha a_\alpha - \bar{A} \tag{4.3}$$

where $\{a_\alpha\}$ corresponds to state vector ψ, $\bar{A}$; see (4.2).

Eigenvalues of operator A are

$$\lambda_{1,2} = 1/2(A \pm R)$$

where $R = [3\bar{A}^2 - 4(A_1A_4 + A_2A_5 + A_3A_6)]^{1/2}$.

A direct calculation gives us the formula for the dispersion $D_\psi A$ of observable A:

$$D_\psi A = 1/2 \sum_\alpha A_\alpha^2 - 3/4A^2 - (\bar{A}/2 - E_\psi A)^2 \tag{4.4}$$

Using the technique described above, consider two spin operators (but without Planck's constant)

$$A = S_x = 1/2\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad B = 1/2\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

Using (4.1), we obtain:

For A: $A_1 = 1/2$; $A_4 = -1/2$; other $A_\alpha = 0$.

For B: $B_2 = 1/2$; $B_5 = -1/2$; other $B_\alpha = 0$.

Now consider an arbitrary state ψ given by its set of weights $\{a_\alpha\}$. In virtue of (4.3) and (4.4) we have

$$E_\psi A = 1/2(a_1 - a_4), \qquad E_\psi B = 1/2(a_2 - a_5)$$

$$D_\psi A = a_1 a_4 \qquad D_\psi B = a_2 a_5$$

The mean value of the observable $S_z = 1/2\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ in state $\{a_\alpha\}$ is $E = 1/2(a_3 - a_6)$. Now consider the expression $\Delta = D_\psi A \cdot D_\psi B - 1/4E^2$. A direct calculation gives $\Delta = a_1a_4a_2a_5 - 1/16(a_3 - a_6)^2$. In virtue of (3.3), Δ is always nonnegative; therefore, in any state ψ

$$D_\psi A \cdot D_\psi B \geq 1/4(E_\psi S_z)^2 \tag{4.5}$$

which is the Heisenberg uncertainty relation without Planck's constant valid for any system described by the graph G (Figure 2) and the corresponding quantum logical lattice.

5. CONCLUDING REMARKS

The correspondence between quantum logical lattices and graphs and negative logics used in the interpretation of states of a system described by graphs leads to the macroscopic realization of quantum logics. Now we can construct classical automata which behave as some quantum system and imitate its features. Here we have investigated only the simplest quantum systems. Nevertheless, we hope that along these lines one can construct automata for more complicated cases. This is very important for computational modeling in quantum theory.

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